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On the relationship between the temperature and the geometry of the two-dimensional superconducting system

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Abstract. A set of vortices in the superconducting system being a two-dimensional region with a boundary has been considered. Here the system under study is described by the model of the Ginzburg-Landau potential in the dual point. This model predicts that in the bounded superconducting system non-interacting vortices appear. These vortices make the absolute minima of this potential. It turned out that in the thermodynamic equilibrium for the fixed number of vortices, the temperature of the system and the geometry of the boundary are related to each other. The simultaneous change of the temperature of the system and of the geometry of the boundary has been investigated under the assumption that the number of vortices is fixed. In the case of the flat disc the explicit form of the temperature vs. area relation has been obtained for two different boundary conditions.

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1 Introduction

The term vortex appears in several domains of physics in two dimensions, e.g. in:

1. superconductivity described by the Ginzburg-Landau (GL) theory,
2. hydrodynamics,
3. gauge theories on lattice.

In general a vortex is an example of topological defect in the physical system. In the case of two dimensions the vortex is a zero-dimensional object. In the superconductors described by the GL theory a vortex is a zero of the function ψ presenting the order parameter. The free energy of superconductors has the form:

$$\mathcal{F}[\psi,\mathbf{A},\kappa]=\int_{\Sigma}d^2x\left[\left|(\boldsymbol{\nabla}-i\mathbf{A})\psi\right|^2+\kappa^2\left(1-|\psi|^2\right)^2+1/2B^2\right],$$

where Σ is the region occupied by the superconducting system, κ is a coupling constant and $B=\varepsilon^{ab}\partial_a A_b$ (magnetic field), where $a,b=1,2$. In the GL theory the assumption is made that in the spatial infinity the asymptotic behavior of ψ is the following:

1. $|\psi|\to 1$,
2. $(\partial_a - iA_a)\psi\to 0$.

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The fields which have the above asymptotic properties are classified by their integer winding number. The winding number n of ψ means that the phase of ψ increases by $2\pi n$ when going anticlockwise around the circle situated at infinity. The second condition on ψ implies that the magnetic flux is quantized:

$$\int_{\Sigma}Bd^2x=2\pi n. \tag{1}$$

In the case, when the zeros of ψ are isolated, the number of zeros counted with their multiplicities is also n. In the vicinity of a zero, which is localized in $\mathbf{x}_0$ with the multiplicity k, the ψ has the form:

$$\psi(\mathbf{x})\simeq a|\mathbf{x}-\mathbf{x}_0|^k.$$

A zero of the multiplicity 1 is called a *vortex* and a zero of multiplicity -1 is called an *antivortex.*

In the dual point $\kappa=1/\sqrt{2}$ the free energy has the form:

$$\mathcal{F}\left[\psi,A,1/\sqrt{2}\right]=\int_{\Sigma}d^2x\left[\left|\overline{\partial}_A\psi\right|^2+\frac{1}{2}\left(B-1+|\psi|^2\right)^2\right]+\int_{\partial\Sigma}[\mathbf{j}+\mathbf{A}]\cdot d\mathbf{l}, \tag{2}$$

where:

$\overline{\partial}_A=\overline{\partial}+1/2iA,\ A=A_1+iA_2$ and $\overline{\partial}=1/2\left(\partial_1+i\partial_2\right)$,
$\mathbf{j}=\mathrm{Im}\left(\psi^*\boldsymbol{\nabla}\psi\right)-|\psi|^2\mathbf{A}$

and $\partial\Sigma$ is the boundary of Σ.

The absolute minimum of $\mathcal{F}$ is given by the following equations called the Bogomolny equations [1] (in the case when $\partial\Sigma$ is an empty set):

$$\overline{\partial}_A \psi = 0, \tag{a}$$
$$B - 1 + |\psi|^2 = 0. \tag{b}$$

The solution ψ of this system of equations has zeros with the positive multiplicities (as it follows from Eq. (a)). Thus in the system there exist only vortices (no antivortices). As it is well-known, one can eliminate the gauge potential A from equation (b) by solving equation (a). In this way one obtains the equation for $|\psi|^2$:

$$\triangle \ln |\psi|^2 + 2\left(|\psi|^2 - 1\right) = 4\pi \sum_{i=1}^{n} \delta\left(\mathbf{x} - \mathbf{x}_i\right), \tag{c}$$

where $\mathbf{x}_i$ means the position of i-th vortex and $\triangle$ is the Laplace operator on Σ. As Taubes showed [2] the solution ψ of equations (a) and (b) exists (modulo gauge transformation) with zeros at n arbitrary points of Σ (in this case Σ is a plane). Thus the value of $\mathcal{F}$ is:

$$\mathcal{F}\left[\psi_v, \mathbf{A}\right] = \int_{\Sigma} B dS,$$

where ψ_v is the solution of both the equation (a) and equation (b). Using the quantization condition on the magnetic field B (Eq. (1)) one can see that the free energy for the system, where n vortices exist, is equal to:

$$\mathcal{F}\left[\psi_v, \mathbf{A}\right] = 2\pi n.$$

The quantization condition expresses the topological properties of the system described by the line bundle L over manifold Σ with the curvature form B (B is the magnetic field). The $\mathcal{F}$ is invariant under the unitary gauge group $U(1)$. Thus the vortices which are related by this gauge group are physically equivalent because they have the same free energy. The set of inequivalent vortices forms a space called the *moduli space* $\mathcal{M}(n)$:

$$\mathcal{M}(n) = \frac{\left\{(\psi, A) : \overline{\partial}_A \psi = 0\ ,\ B - 1 + |\psi|^2 = 0 \text{ and } \int_{\Sigma} B dS = 2\pi n\right\}}{\left\{u : \Sigma \to U(1)\right\}}.$$

The moduli space $\mathcal{M}(n)$ is isomorphic with the space

$$\frac{\overbrace{\Sigma \times ... \times \Sigma}^{n}}{\Pi(n)},$$

where $\Pi(n)$ is the permutation group acting on the n zeros of ψ.

In the case when the boundary of Σ is not empty the last term in (2) has the form:

$$\int_{\partial\Sigma} [\mathbf{j} + \mathbf{A}] \cdot d\mathbf{l} = \int_{\Sigma} \left[\psi^* \omega + d\left(\mu(\psi) A\right)\right], \tag{3}$$

where ω is a symplectic form on the complex space $\mathbf{C}^1$:

$$\omega = \frac{i}{2} dz \wedge d\overline{z},$$

$\mu(\psi) = 1 - |\psi|^2$ and ψ is a mapping of Σ into $\mathbf{C}^1$; in other words ψ is a section of the line bundle L over Σ. The general form of the energy functionals similar to $\mathcal{F}$ was considered in [3] where it is shown a.o. that the equation (3) is a topological invariant. Under a gauge transformation

$$\psi \to e^{i\lambda}\psi,$$
$$A \to A + id\lambda,$$

the free energy (2) remains invariant. Under diffeomorphisms which conserve the area of Σ equation (2) also remains invariant.

2 Vortices in the finite size systems

As it is well-known, vortices do not interact with each other at the dual point of a finite superconducting system with boundaries. They are being repelled from the boundaries by the edge currents. At the thermodynamic equilibrium all vortices collapse into a one vortex state. The free energy $\mathcal{F}_S$ for such a kind of system has been calculated in [4]. The value of $\mathcal{F}_S$ for n vortices is equal to:

$$\mathcal{F}_S = 2\pi n + \oint_{\partial\Sigma} f\left(k_g\right) dl, \tag{4}$$

where $f\left(k_g\right)$ is the function of the geodesic curvature of the boundary $\partial\Sigma$.

Let us consider the thermodynamic equilibrium of a finite size bounded system of vortices. Then the free energy $\mathcal{F}_V$ for this system is:

$$\mathcal{F}_V = U - TS,$$

where U is the internal energy, S is entropy and T is temperature of the system in the equilibrium. Thus the set of vortices in the superconducting system will be in the thermodynamic equilibrium if:

$$\mathcal{F}_S = \mathcal{F}_V.$$

It means that:

$$2\pi n + \oint_{\partial\Sigma} f\left(k_g\right) dl = U - TS. \tag{5}$$

To determine U the model, in which each vortex is a source of the quantum magnetic flux and all the vortices form the Coulomb gas, is used. Thus U assumes the form:

$$U = 2\pi J \sum_{\mathbf{x},\mathbf{x}'} n(\mathbf{x}) G\left(\mathbf{x} - \mathbf{x}'\right) n\left(\mathbf{x}'\right),$$

where $n(\mathbf{x})$ is a wave function of a vortex in the position $\mathbf{x}$, $G\left(\mathbf{x} - \mathbf{x}'\right)$ is given by:

$$G\left(\mathbf{x} - \mathbf{x}'\right) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dk_1 dk_2 \frac{e^{i\mathbf{k}\left(\mathbf{x} - \mathbf{x}'\right)}}{2 - 2\cos k_1 - 2\cos k_2}.$$

In the case under consideration all vortices are localized in the geometric center of $\varSigma$ and the internal energy is equal to:

$$U = 2\pi J G(0)\, n^2$$

(n is the number of vortices in the system). $2\pi JG(0)$ is the energy of a vortex. This energy is estimated as:

$$2\pi J G(0) = \pi J \ln(R_s/a),$$

where R_s is the linear size of the sample and a is a cut-off parameter which makes the smallest scale where the continuous model is valid. The entropy S for this system is given by the logarithm of the volume of the moduli space $\mathcal{M}(n)$. Such a volume was derived in [5] for the systems existing on any Riemannian surface with the genus g without boundaries. For the system with the boundary the solutions of equation (c) exist only for correctly chosen boundary conditions. Thus the moduli space and its volume depend on these boundary conditions. We assume that such boundary conditions exist (e.g. the Dirichlet boundary conditions in [6]). A boundary $\partial\mathcal{M}_b(n)$ of the moduli space $\mathcal{M}_b(n)$ of n vortices on two-dimensional surface $\varSigma$ has the form:

$$\partial\mathcal{M}_b(n) = \frac{\{n\cdot(\cup_i C_i)\}\times \overbrace{\varSigma\times\ldots\times\varSigma}^{n-1}}{\varPi(n)},$$

where $\cup_i C_i$ is the boundary of $\varSigma$. In the case of the collapsing n vortices the moduli space is one-dimensional (the complex dimension) and the volume of the moduli space assumes the form [5]:

$$vol(\mathcal{M}(n)) = nA + n\int_{\varSigma} db, \tag{6}$$

where 2-form db is given by:

$$db = (\partial_j b_i - \partial_i b_j)\, dx^i \wedge dx^j.$$

The functions b_i are determined from equation (c). For boundless surface $\varSigma$ with genus g the integral from db is given in [5] and is equal to $4\pi n(g-1)$ (in [5] is also considered a more general case when the vortices do not collapse). Thus the volume of the moduli space in the boundless case has the form:

$$vol(\mathcal{M}) = An - 4\pi n^2(1-g).$$

If surface $\varSigma$ has the boundary $\partial\varSigma$, then the equation (6) assumes the following form:

$$vol(\mathcal{M}_b(n)) = nA + n\int_{\partial\varSigma} b_i dx^i,$$

where the functions b_i depend on the boundary conditions and on the relative positions of the vortices. These positions are collectively denoted as ρ. In [6] is considered the case of a one vortex on a disk with Dirichlet boundary conditions: $|\psi|^2 = 1$ on $\partial\varSigma$. The volume of the moduli space for such a system depends on the distance ε between the boundary and localization of the vortex:

$$vol(\mathcal{M}_b(1)) = A + 2\pi R/\varepsilon, \tag{7}$$

where A is the area of the disk with radius R.

Thus the volume of the moduli space, for any boundary conditions, is represented as follows:

$$vol(\mathcal{M}_b(n)) = nA + nZ(\rho, \partial\varSigma),$$

where

$$Z(\rho, \partial\varSigma) = \int_{\partial\varSigma} b_i dx^i. \tag{8}$$

In general, $Z(\rho, \partial\varSigma)$ depends on: the number of vortices n, the parameters describing the boundary (these parameters are represented also by $\partial\varSigma$) and on the boundary conditions put on ψ. In this way one can see that the entropy for the system with boundary is:

$$S = k\ln[An + nZ(\rho, \partial\varSigma)], \tag{9}$$

where A is the area of $\varSigma$. One can now express the equation (5) as follows:

$$2\pi n + \oint_{\partial\varSigma} f(k_g)\, dl = 2\pi JG(0)\, n^2 - kT\ln[An + nZ(\rho, \partial\varSigma)]. \tag{10}$$

Thus in the thermodynamic equilibrium of the superconducting system the three parameters: the number of vortices n, the temperature T and the form of the boundary $\partial\varSigma$ turned out to be related to each other.

Let us introduce the following function $\mathcal{G}$:

$$\mathcal{G}(n, T; \partial\varSigma) = 2\pi n - 2\pi JG(0)\, n^2 + \oint_{\partial\varSigma} f(k_g)\, dl + kT\ln[An + nZ(\rho, \partial\varSigma)].$$

One can then say that the set of vortices appearing in the superconducting system is in thermodynamic equilibrium if the function $\mathcal{G}$ vanishes:

$$\mathcal{G}(n, T; \partial\varSigma) = 0. \tag{11}$$

In the next section we consider how the change of the boundary leads to the change of temperature of the system when the above condition is obeyed (what means that the system remains in the superconducting state).

3 Critical temperatures and the temperature vs. area relation

Let us consider the case when $\mathcal{F}_S$ is equal to zero. Thus the equation (5) leads to the condition that:

$$\mathcal{F}_V = U - TS = 0.$$

From this equation one finds how does the temperature depend on the number of vortices. If the explicit form of the function f in equation (4) is known, then one can in principle change the shape of the boundary (in other words: of the geodesic curvature) in such a way that the free energy $\mathcal{F}_S$ vanishes. It is the way to determine the number of vortices in the system. If $\mathcal{F}_V = 0$ then the temperature of the set of vortices is equal to the critical temperature T_c (note that the superconducting critical temperature is denoted by T_S). Then T_c is expressed by:

$$T_c(n, A) = \frac{2\pi JG(0)\, n^2}{k \ln\left[An + nZ(\rho, \partial\Sigma)\right]}. \tag{12}$$

From equation (9) it follows that n should be such that: $An + nZ(\rho, \partial\Sigma) > 0$. For the boundless case [5], i.e. for $\partial\Sigma = \varnothing$,

$$Z(r, \varnothing) = 4\pi n(g-1)$$

for $g \geq 1$, the number n of vortices is unbounded since the argument of the above logarithm is always positive ($An > 0$). For $g = 0$ (Σ is a 2-dimensional sphere) this condition becomes the Bradlow inequality [7]: $4\pi n \leq A$. Thus the number of vortices for $T = T_c$ is related to the shape of the boundary of Σ by (see Eq. (5)):

$$2\pi n = -\oint_{\partial\Sigma} f(k_g)\, dl. \tag{13}$$

Since in the system only vortices (but not antivortices) exist, the above condition means that:

$$\oint_{\partial\Sigma} f(k_g) < 0.$$

In the case when k_g is constant the above inequality reduces to:

$$f(k_g) < 0.$$

It is easy to see that for a temperature T below $T_c(n, A)$ there exist exactly n vortices for fixed both the area A and the shape of boundary, because the sequence of critical temperatures $T_c(n, A)$ increases with n. The latter relation is expressed by the following inequality:

$$T_c(n+1, A) > T_c(n, A).$$

It means physically that the magnetic field inside the system increases destroying the superconducting phase. This effect is equivalent to the increase of temperature of the system. The sequence $\{T_c(n, A)\}$ is upper bounded by the critical temperature of the superconducting phase transition T_S:

$$T_S \geq ... \geq T_c(n+1, A) \geq T_c(n, A) \geq ...$$

If these critical temperatures become higher than T_S then the superconducting phase vanishes. Thus the maximal number of vortices which exist in the superconducting phase can be found from the following equation:

$$T_c(n_{\max}, A) = T_S, \tag{14}$$

where $n_{\max}$ is the maximal number of vortices in the superconducting system. Thus the equation (12) assumes the form:

$$An + nZ(r, \partial\Sigma) = \exp\left[\beta_S 2\pi JG(0)\, n^2\right],$$

where $\beta_S = 1/kT_S$. Solving the above equation with respect to n and using equation (13) one obtains the following relation:

$$n_{\max}(T_S) = -\oint_{\partial\Sigma} f(k_g)\, dl, \tag{15}$$

which shows that number of vortices $n_{\max}$ is related to T_S and the shape of the boundary of the region in which the system is placed.

In order to estimate the change temperature T of the system with the change of the boundary one has to know the explicit form of the function$f$. Comparing equations (2), (3) and (4) one can notice that:

$$f(k_g) = \left(1 - |\psi|^2\right) \frac{A_i \dot{x}^i}{\left|\dot{x}\right|}, \tag{16}$$

where ψ and A are solutions of equations (a) and (b). These solutions are restricted to the boundary $\partial\Sigma$, which is parameterized by $x^i = x^i(s)$, $i = 1, 2$ and s is a parameter. The unit vector $\mathbf{t}$ tangent to $\partial\Sigma$ is:

$$\mathbf{t} = \frac{1}{\left|\dot{x}\right|}\left(\dot{x}^1, \dot{x}^2\right).$$

Thus the function f depends on the tangent component A_t of the vector field $\mathbf{A}$:

$$f(k_g) = \left(1 - |\psi|^2\right) A_t,$$

where: $A_t = \mathbf{A} \cdot \mathbf{t}$. In this way the function $\mathcal{G}$ takes the form:

$$\mathcal{G}(n, T; \partial\Sigma) = 2\pi n - 2\pi JG(0)\, n^2 + \oint_{\partial\Sigma} \left(1 - |\psi|^2\right) A_t dl + kT \ln\left[An + nZ(\rho, \partial\Sigma)\right] \tag{17}$$

Let us estimate the change of the function $\mathcal{G}$ under the deformation of the boundary $\partial\Sigma$. In this aim we consider the variation of the integral I:

$$I = \oint_{\partial\Sigma} \theta_i dx^i,$$

where:

$$\theta_i = \left(1 - |\psi|^2\right) A_i.$$

Let the deformation of $\partial\Sigma$ be given by a vector field X:

$$X = X^i(x) \frac{\partial}{\partial x^i}.$$

Thus the variation of I under X has the form:

$$\delta_X I = \oint_{\partial\Sigma} L_X\left(\theta_i dx^i\right) = \oint_{\partial\Sigma}\left[X^i(x)F_{ij}dx^j + d\left(X^i\theta_i\right)\right] \tag{18}$$

where:

$$F_{ij} = \partial_i\theta_j - \partial_j\theta_i,$$

and L_X is a Lie derivative along the vector field X. In two dimensions δI has the form:

$$\delta_X I = \oint_{\partial\Sigma}\left[F\mathbf{X}\times d\mathbf{x} + d\left(X^i\theta_i\right)\right], \tag{19}$$

where: $F = \partial_1\theta_2 - \partial_2\theta_1$. For a given parameterization $\mathbf{x} = \mathbf{x}(t)$ of the boundary the first term in equation (19) can be reformulated as follows:

$$\oint_{\partial\Sigma} F\mathbf{X}\times\dot{\mathbf{x}}dt.$$

It means that the perpendicular component $\mathbf{X}_v$ of the vector $\mathbf{X}$ with respect to the tangent vector $\dot{\mathbf{x}}$ gives non-zero contribution to the integral. Let us decompose the vector $\mathbf{X}$ into the orthonormal basis of two vectors $\mathbf{t}$ and $\mathbf{n}$, the former vector being tangent and the latter being normal to the boundary:

$$\mathbf{X} = X_h\mathbf{t} + X_v\mathbf{n}.$$

The vectors $\mathbf{t}$ and $\mathbf{n}$ are related with each other as follows:

$$\frac{d\mathbf{t}}{dt} = \kappa\mathbf{n},$$

where κ is the curvature of the boundary and $\mathbf{t} = \dot{\mathbf{x}}/\left|\dot{\mathbf{x}}\right|$. In this way one obtains that:

$$\oint_{\partial\Sigma} F\mathbf{X}\times\dot{\mathbf{x}}dt = \oint_{\partial\Sigma} F\left(\mathbf{X}\cdot\mathbf{n}\right)\left|\dot{\mathbf{x}}\right| dt.$$

The second term in (19) can be interpreted as a change in the number of zeros and poles of the function $\exp\left(X^i\theta_i\right)$. These zeros and poles are related both to the creation or annihilation of vortices in the system when the boundary changes.

For the case when Σ is a disc with a radius R on a 2-dimensional plane $\mathbf{R}^2$ the boundary is a circle with the radius R. The entropy of the system in this case ($g = 0$) is equal to:

$$S = k\ln\left[An + nZ\left(\rho, \partial\Sigma\right)\right],$$

where $A = \pi R^2$. The function Z (see Eq. (8)) in the polar coordinates (r, φ) takes the form:

$$Z\left(\rho, \partial\Sigma\right) = R\int_0^{2\pi} b_\varphi\left(R, t; \rho\right) dt. \tag{20}$$

For this system the orthonormal basis is given by:

$$\mathbf{n} = \frac{\partial}{\partial r},$$
$$\mathbf{t} = \frac{\partial}{r\partial\varphi}.$$

Any vector field X is expressed in the above basis as follows:

$$X = \varepsilon\frac{\partial}{\partial r} + \delta\frac{\partial}{r\partial\varphi},$$

where ε is related to the change of the radius of the disc and δ is related to the reparametrization of the circle. Next we will consider the coefficients ε and δ which are independent of the points $\mathbf{x}$ of the disk. Since $\left|\dot{\mathbf{x}}\right| = R$ the variation of the integral has the form:

$$\oint_{\partial\Sigma}\mathbf{X}\cdot\mathbf{n}\left|\dot{\mathbf{x}}\right| F dt = \varepsilon R\oint_{\partial\Sigma} F dt.$$

Thus $\delta_X I$ is equal to:

$$\delta_X I = \varepsilon R\oint_{\partial\Sigma} F dt + \varepsilon\left(\theta_r(2\pi) - \theta_r(0)\right) + \delta\left(\theta_\varphi(2\pi) - \theta_\varphi(0)\right).$$

In the case when θ_r and θ_φ are periodic (which means that no vortex appeared or vanished in the deformation process) the δI reduces to:

$$\delta_X I = \varepsilon R\oint_{\partial\Sigma} F dt.$$

The coefficient εR is proportional to the change of the area $\delta_X A$:

$$\delta_X A = 2\pi\varepsilon R.$$

Under the deformation of the boundary $\delta\partial\Sigma$ and the change of the temperature δT the function $\mathcal{G}$ takes the form:

$$\mathcal{G}\left(n, T + \delta T; \partial\Sigma + \delta\partial\Sigma\right) = \mathcal{G}\left(n, T; \partial\Sigma\right) + \delta T\delta_T\mathcal{G}\left(n, T; \partial\Sigma\right) + \left(\delta\partial\Sigma\right)\delta_{\partial\Sigma}\mathcal{G}\left(n, T; \partial\Sigma\right),$$

where $\delta\partial\Sigma = \delta A/2\pi$. The system stays in the superconducting phase if:

$$\mathcal{G}\left(n, T + \delta T; \partial\Sigma + \delta\partial\Sigma\right) = 0. \tag{21}$$

Because the entropy depends on the area of the disk and the parameters related to the boundary one obtains the following variations of $\mathcal{G}$ with respect to the temperature T and the vector field X:

$$\delta_T\mathcal{G}\left(n, T; \partial\Sigma\right) = S\delta T,$$
$$\delta_{\partial\Sigma}\mathcal{G}\left(n, T; \partial\Sigma\right) = T\left(\frac{\partial S}{\partial A}\delta_X A + \frac{\partial S}{\partial Z}\delta_X Z\right) + \delta_X I,$$

where $S = k\ln\left(An + nZ\left(r, \partial\Sigma\right)\right)$. Since $\mathcal{G}\left(n, T; \partial\Sigma\right) = 0$ one obtains from equation (21) the following relation:

$$\delta TS + T\left(\frac{\partial S}{\partial A}\delta_X A + \frac{\partial S}{\partial Z}\delta_X Z\right) + \delta_X I = 0. \tag{22}$$

The variation $\delta_X Z$ (which is analogously obtained as $\delta_X I$) is equal to:

$$\delta_X Z = \oint_{\partial\Sigma}\left[u\mathbf{X}\times d\mathbf{x} + d\left(X^i b_i\right)\right],$$

where $u = \partial_1 b_2 - \partial_2 b_1$. Thus:

$$\delta_X Z = \varepsilon R \oint_{\partial\Sigma} u dt + \varepsilon \left(b_r(2\pi) - b_r(0)\right) + \delta\left(b_\varphi(2\pi) - b_\varphi(0)\right).$$

Assuming periodicity functions b_r and b_φ on the boundary we obtain:

$$\delta_X Z = \varepsilon R \oint_{\partial\Sigma} u dt.$$

In the polar coordinates (r, φ) the functions F and u have the form:

$$F = \frac{1}{r}\left[\frac{\partial(r\theta_\varphi)}{\partial r} - \frac{\partial\theta_r}{\partial\varphi}\right],$$
$$u = \frac{1}{r}\left[\frac{\partial(rb_\varphi)}{\partial r} - \frac{\partial b_r}{\partial\varphi}\right].$$

Integrating F and u over the circle $\partial\Sigma$one obtains:

$$\oint_{\partial\Sigma} F dt = \frac{1}{R}\frac{\partial}{\partial r}\left[r\int_0^{2\pi}\theta_\varphi(r,t)\,dt\right]|_{r=R}, \qquad (23)$$
$$\oint_{\partial\Sigma} u dt = \frac{1}{R}\frac{\partial}{\partial r}\left[r\int_0^{2\pi} b_\varphi(r,t)\,dt\right]|_{r=R}, \qquad (24)$$

where

$$\theta_\varphi(r,t) = \left(1 - |\psi(r,t)|^2\right) A_\varphi(r,t).$$

Therefore, from equation (22) one obtains the following relation:

$$\delta T S + T\left(\frac{\partial S}{\partial A} + \frac{1}{2\pi}\frac{\partial S}{\partial Z}K\right)\delta A + \frac{1}{2\pi}W\delta A = 0,$$

where:

$$K(A) = \frac{1}{R}\frac{\partial}{\partial r}\left[r\int_0^{2\pi} b_\varphi(r,t)\,dt\right]|_{r=R},$$
$$W(A) = \frac{1}{R}\frac{\partial}{\partial r}\left[r\int_0^{2\pi}\theta_\varphi(r,t)\,dt\right]|_{r=R}.$$

Note that in the above formulas the relation $\delta A/2\pi = \varepsilon R$ was used. In this way the following differential equation on T has been obtained:

$$\frac{dT}{dA}S + T\frac{dS}{dA} + T\frac{1}{2\pi}\frac{\partial S}{\partial Z}K + \frac{1}{2\pi}W(A) = 0 \qquad (25)$$

with the initial condition $T(A_0) = T_0$, where T_0 is the initial temperature for the area A_0. The equation (25) can be rewritten as follows:

$$\frac{d(TS)}{dA} + T\frac{1}{2\pi}\frac{\partial S}{\partial Z}K + \frac{1}{2\pi}W(A) = 0. \qquad (26)$$

The general solution of equation (26) has the form:

$$T(A) = T(A_0)\frac{S(A_0)}{S(A)}\exp\left(-\int_{A_0}^{A}\frac{\partial \ln S}{\partial Z}K(a)\,da\right) \times\left[1 - \frac{1}{2\pi T(A_0)S(A_0)} \times\int_{A_0}^{A}\exp\left(\int_{A_0}^{a}\frac{\partial\ln S}{\partial Z}K(c)\,dc\right)W(a)\,da\right]. \qquad (27)$$

Note that in the case under consideration the following relation is valid: $\sqrt{A/\pi} = R$. From equation (20) one can see that:

$$K(A) = \frac{1}{R}\frac{\partial Z}{\partial R}.$$

Let us consider the boundary conditions that lead to Z, which is independent of R. Such boundary conditions are realized when in equation (20) the angular part of the 1-form b has the form:

$$b_\varphi(R,t;\rho) = \frac{1}{R}\widetilde{b}_\varphi(t;\rho).$$

We put the following boundary conditions on $\ln|\psi|^2$:

$$\partial_t \ln|\psi|^2|_{r=R} = 0.$$

In the case considered all the vortices are localized in the center of the disk. Thus near this center the following expansion [5] in the polar coordinates r, t on the disc is valid:

$$\ln|\psi|^2 = \ln r + a + rb_\varphi(r,t;\rho) + ...,$$

(we use the same polar coordinates both on the moduli space of vortices and on the disk). The boundary conditions give the following restriction on b_φ:

$$b_\varphi(R,t;\rho) = \widehat{b}_\varphi(R;\rho).$$

Thus b_φ should has the form:

$$b_\varphi(R,t;\rho) = \frac{1}{R}\widetilde{b}_\varphi(\rho).$$

It means that $\widetilde{b}_\varphi$ depends only on ρ (for $r = R$). In this case $K = 0$. Thus equation (27) has the form:

$$T(A) = \frac{1}{k\ln[An + Zn]} \times\left(T_0 k\ln[A_0 n + Zn] - \frac{1}{2\pi}\int_{A_0}^{A} W(a)\,da\right). \qquad (28)$$

Taking into account the equation (b) one obtains that:

$$W(A) = \frac{1}{R}\frac{\partial}{\partial R}\left[RB(R)J(R)\right]|_{R=\sqrt{A/\pi}},$$

where:

$$J(R) = \int_0^{2\pi} A_\varphi(R,t)\,dt.$$

In the disc case the area element is: $da = 2\pi r dr$. Thus equation (28) has the form:

$$T(A) = \frac{1}{k\ln[An + Zn]} \times\left(T_0 k\ln[A_0 n + Zn] - \int_{R_0}^{R}\frac{1}{r}\frac{\partial}{\partial r}\left[rB(r)J(r)\right]rdr\right).$$

$$T(A) = T(A_0)\frac{S(A_0)}{S(A)}\exp\left(2n\rho\int_{\sqrt{A_0/\pi}}^{\sqrt{A/\pi}} dr \frac{1}{\left(r^2(r-\rho)+2rn\right)\ln\left(n\pi r^2 + \frac{2\pi r n^2}{r-\rho}\right)}\right).$$

The integration in the last formula is simple, because the function under the integral makes a full derivative. Thus one obtains:

$$T(A) = \frac{1}{k\ln[An+Zn]}\left[T_0 k \ln[A_0 n + Zn] - \left(RB(R)J(R) - R_0 B(R_0) J(R_0)\right)\right].$$

Let us estimate the last formula in the case of the angular gauge field A_φ, which scales like $A_\varphi \sim 1/r$. Hence the gauge strength field B scales like $1/r^2$. Thus:

$$RB(R)J(R) \sim R\frac{1}{R^2}\frac{1}{R} \sim 1/A,$$

and:

$$T(A) = T_0\frac{\ln[A_0 n + Zn]}{\ln[An+Zn]} - \frac{1}{k\ln[An+Zn]}\left(1/A - 1/A_0\right). \tag{29}$$

Thus the law is obtained of the change of temperature of the set of vortices with the change of the area for the superconducting phase.

For the Dirichlet boundary condition, $|\psi| = 1$ on the boundary, one obtains:

$$W = 0$$

and (from Eq. (7))

$$Z = \frac{2\pi R n}{R-\rho}$$

where ρ is a distance between the center of the disk and the position of the collapsed vortices. Let us consider that the vortices collapse in the center of the disk which means that $\rho = 0$. Thus the entropy is:

$$S = \ln\left(nA + \frac{2\pi R n^2}{R-\rho}\right),$$

and:

$$K = -\frac{2\pi\rho n}{(R-\rho)R}.$$

Equation (27) assumes the form:

See the equation above.

For $\rho = 0$ one obtains a simple relation:

$$T(A) = T(A_0)\frac{\ln\left(nA_0 + 2\pi n^2\right)}{\ln\left(nA + 2\pi n^2\right)}. \tag{30}$$

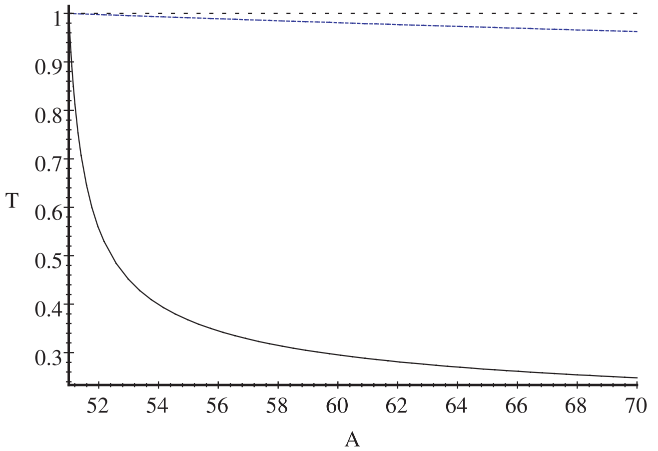

Fig. 1. Temperature T vs. area A. Dotted line corresponds to $T = 1$, solid line corresponds to equation (29) for $n = 4$, dashed line corresponds to equation (30) for $n = 4$.

This relation means that the product of the temperature T and the entropy S is constant when the area A changes:

$$T(A)S(A) = \text{const}.$$

One can then see that the different boundary conditions lead to the different relations between temperature and entropy. This conclusion reflects the fact that the conditions on the boundary of the system influence the system in the bulk. Let us compare the diagrams based on equation (29) and equation (30) and scaled by $T(A_0) = 1$ and $k = 1$. In equation (29) we assume that Z is proportional to n; note that the proportionality coefficient has been chosen in such a way that it agrees with the sphere case:

$$Z = -4\pi n.$$

Let us assume that the number n of vortices is equal to 4. From the Bradlow inequality one obtains the constraint on A: $A > 4 \times 4\pi = 50.265$. Let us choose for simplicity that $A_0 = 51$.

One can see from Figure 1 that for the fixed number of vortices the T vs. A relationship strongly depends on the boundary conditions.

Figure 2 presents $T(A)$ for one vortex ($n = 1$). As it follows from the Bradlow inequality, $A > 4\pi = 12.566$. Therefore the initial value of the area is chosen as before: $A_0 = 51$, $T(A_0) = 1$.

As it follows from the comparison of Figure 1 and Figure 2 the lower the number n of vortices the lower the sensitivity of the area dependence of temperature on the boundary conditions.

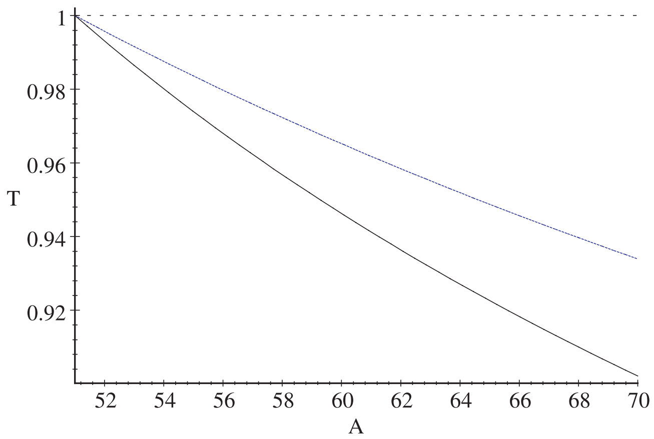

Fig. 2. Temperature T vs. area A. Dotted line corresponds to $T = 1$, solid line corresponds to equation (29) for $n = 1$, dashed line corresponds to equation (30) for $n = 1$.

4 Conclusions

A sequence of critical temperatures related to the number of vortices which appear at the dual point of a superconductor has been obtained. This sequence is limited from the top by the temperature of the superconducting phase transition. Applying the latter condition one can evaluate the maximal number of vortices existing in the superconducting phase. Equation (27) gives the explicit relation between the area of a disc and the temperature of the two-dimensional system. This relation strongly depends on the boundary conditions put on ψ, since solutions of the equation (c) exist only for the appropriate boundary value problems (bvp). Moreover the entropy of the system depends on bvp and the selection of bvp is crucial for the behavior of the system. The Dirichlet boundary condition on ψ has been considered in [6]. The study of the other bvp is also interesting, e.g. the Neumann or mixed boundary value problems.

It is interesting to investigate the bounded two-dimensional superconducting systems on the surfaces with the constant non-zero curvature, e.g. on a sphere which has the positive curvature and on a hyperbolic plane which has the negative curvature. In these cases one should have obtained the analogous equations to equation (27). In the sphere case when the system is bounded by a circle laying on this sphere the dependence of the temperature on the area should be similar as in the flat case considered above. In the hyperbolic plane case we assume that the vortices move on the geodesics. From the negative constant curvature of the hyperbolic plane it follows that the two vortices, which are close to each other at the beginning, run after a finite time far away from each other. However these vortices can not reach the boundary itself because of the repulsive edge currents appearing on it. Finally they form near the boundary a structure which could present a kind of the image of the boundary. In order to proof the above hypothesis further investigations are needed.

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